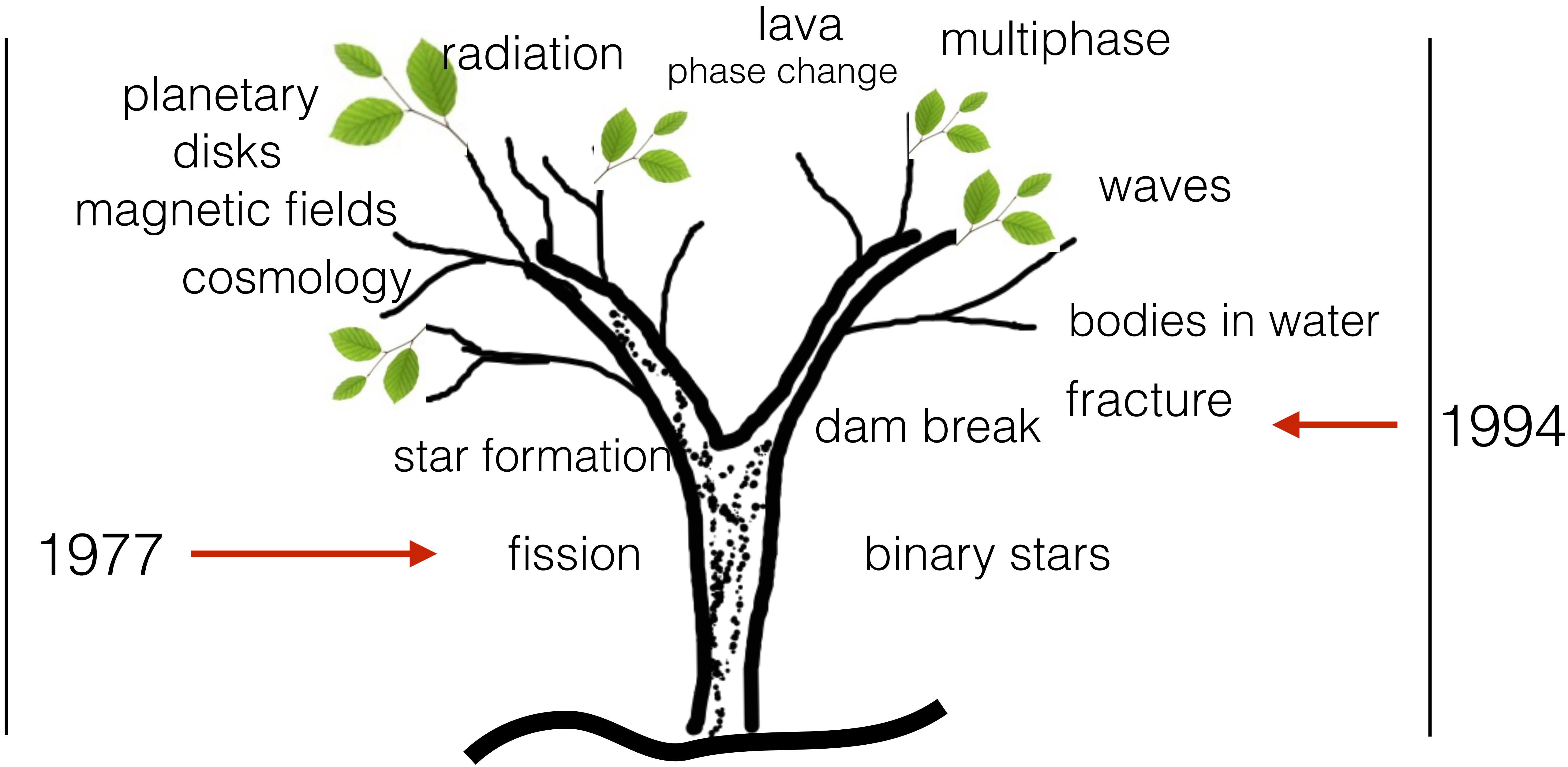
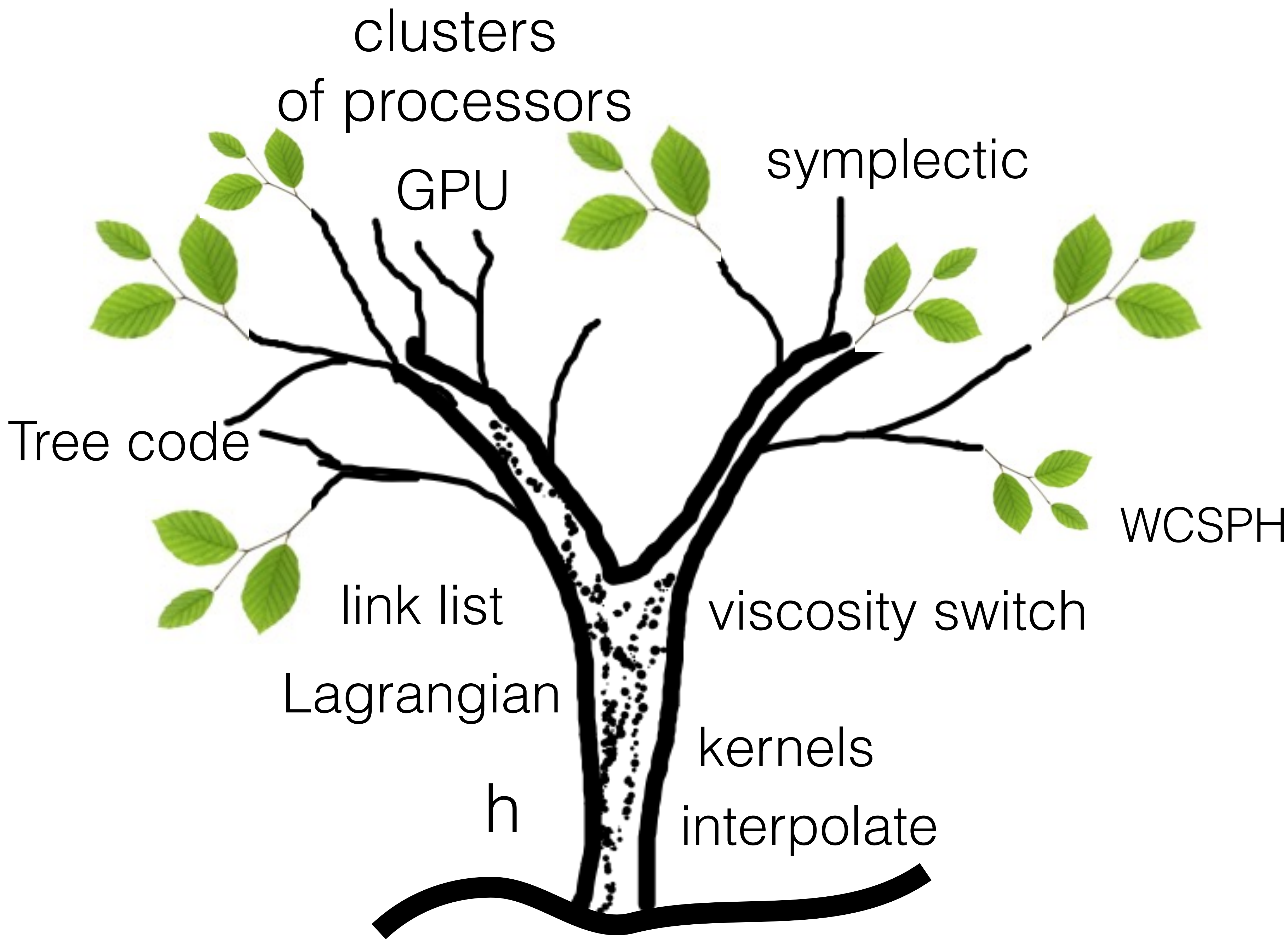
The Evolution of SPH

J. J. Monaghan
Monash University
Australia



SPH tree of life



SPH tree of wisdom

The aim in 1976 was
to simulate star formation

which occurred in a complicated way
in complicated regions.

The only available techniques were finite difference and they were complicated for star formation problems.

Bob Gingold and I decided to try a particle method.

First SPH paper by RAG and JJM evolved models to a static state.

The code did not conserve momentum

Mon. Not. R. astr. Soc. (1977) **181**, 375–389

Smoothed particle hydrodynamics: theory and application to non-spherical stars

R. A. Gingold and J. J. Monaghan* *Institute of Astronomy, Madingley Road, Cambridge, CB3 0HA*

A related paper by Leon Lucy discussed fission of stars. The algorithm did not conserve momentum.

THE ASTRONOMICAL JOURNAL VOLUME 82, NUMBER 12 DECEMBER 1977

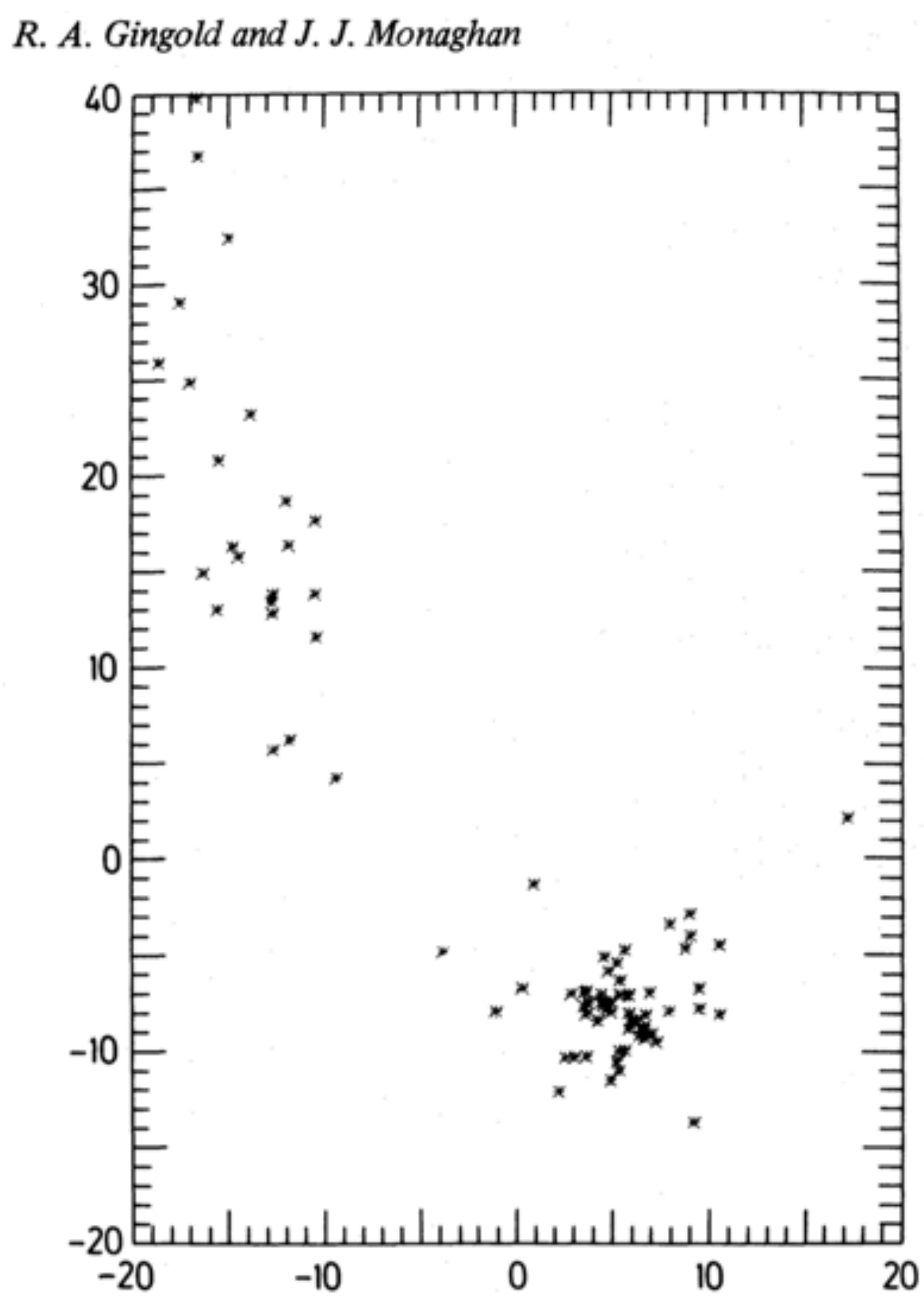
A numerical approach to the testing of the fission hypothesis

L. B. Lucy[a)]
Institute of Astronomy, Cambridge, United Kingdom
European Southern Observatory, Geneva, Switzerland
(Received 12 August 1977; revised 16 September 1977)

A finite-size particle scheme for the numerical solution of two- and three-dimensional gas dynamical problems of astronomical interest is described and tested. The scheme is then applied to the fission problem for optically thick protostars. Results are given, showing the evolution of one such protostar from an initial state as a single, rotating star to a final state as a triple system whose components contain 60% of the original mass. The decisiveness of this numerical test of the fission hypothesis and its relevance to observed binaries are briefly discussed.

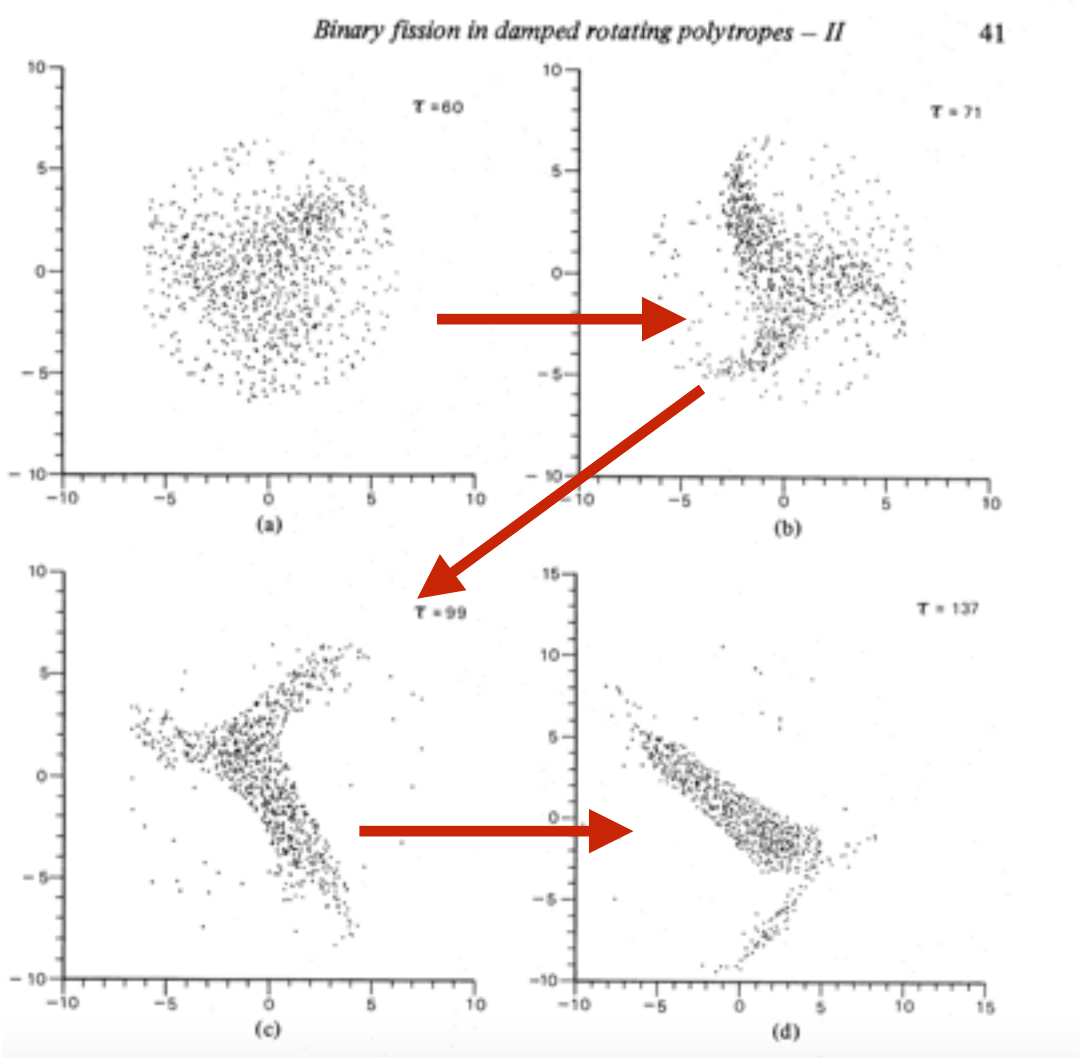
300 particles used

RAG AND JJM
fission of a rotating gas cloud using 80 SPH particles and equations based on a Lagrangian in 1978



Fission with 800 SPH particles in 1979

Higher resolution gives higher accuracy



But this was far removed from
current astrophysical
SPH calculations

How did we get from
the low accuracy of 1979 to
the high accuracy of 2015 for
astrophysical problems?

Many of the SPH algorithms now used can be found by elegant methods that now seem obvious.

But the way you find things out is to try many things for simple cases until the correct form is found.

Then you often cover up the hard work.

Some of this work is in my
notebooks

Here are some of the topics considered in the first notebook.

Index   Begun around late Sept 1976

date 1976

1) Smoothing and estimating
3) Criterion of fit   1D
5) 3D results
6) Spherical spline
13) Grav potential
16) Equs of motion
17) Conservation of momentum
18) Ang. Momentum
19) Virial
(20) Energy
23) Polytrope of index n
26) Introduction of uniform rotation.
27) Use of spline for polytrope
29) Energy and angular momentum in the rotating frame
33) Magnetic field smoothing 1D and Alfvén wave test program
(36) Numerical calculations - perturbation method & constant.
(38) Polytrope + mag. field
(39) 3D field smoothing
(41) Starting field.
(44) Importance sampling for $\vec{B}$
(49) Finding $d\vec{B}/dt$.
(50) Flux freezing check.
(52) Checking the field
(57) Remarks on choosing f for the magnetic field
(58) Variance
(59) No of points in $r \to r+dr$ polytrope.
(60) B_x for dipole field
(61) Mass fraction for differential rotation
(62) Estimating K.E.

Equ. of motion

B smoothing

f = 1/h**2

(76) Some forms of f.
(81) Introduction of f depending on grav. energy.

(87) Estimate of fluctuation of the potential.
(89) Some suggestions for reducing N^2 factor
(90) Errors in other integration methods
(91) Remarks on secular instability
(94) Application to potential problems – Q.M.
(95) A super smoothing function which removes bias.

(103) $$\iint \frac{e^{-a(\underline{r}-\underline{r}_a)^2-b(\underline{r}'-\underline{r}_b)^2}}{|\underline{r}-\underline{r}'|}\,d\underline{r}\,d\underline{r}'$$

Smoothed Particle (SP) Hydrodynamics

The method uses a Lagrangian picture of fluid motion. Th
positions of elements of fluid are taken to specify the
density distribution. The density is recovered from the
positions by using the methods employed by statisticians

E. Parzen.
Ann. Math. Stats
33, 1065, (

Boneva, Kendall,
Stefanov
J. Roy Stat Soc
,

Smoothing and Estimating Distributions

One Dimension

$$g(x) = \int \delta(x-y)\, g(y)\, dy$$

Let the estimator be

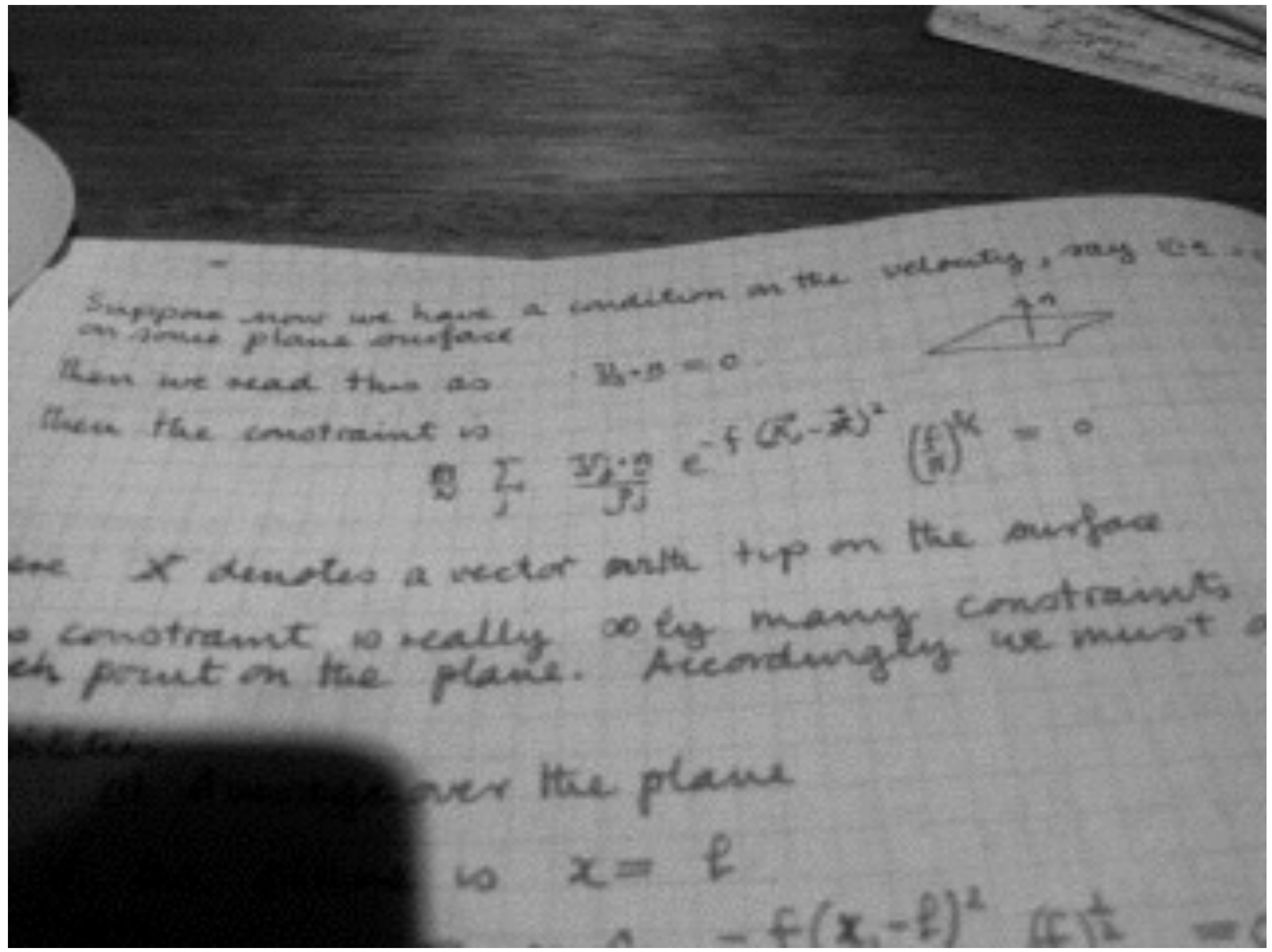
$$g_n(x) = \frac{1}{n} \sum_{j=1}^{n} W(x-x_j) \quad \int g(y)\, dy.$$

which is equivalent to

$$E\left(\frac{1}{n} \sum_{j=1}^{n} W(x-x_j) \right) = \frac{\int W(x-y)\, g(y)\, dy}{\int g(y)\, dy}.$$

If W is strongly peaked then we get a good estimate of g(

Implementing a boundary condition by a Lagrangian multiplier constraint



$\mathbf{v} \cdot \mathbf{n} = 0$ boundary condition

$$\sum_j \frac{\mathbf{v}_j \cdot \mathbf{n}_j}{\rho_j} \left(\frac{f}{\pi}\right)^{3/2} e^{-f(\mathbf{x}-\mathbf{x}_j)^2} = 0.$$

Lagrangian used by RAG and JJM in 1978

kinetic energy

pressure energy

gravitational energy

spherical polytrope. Using the new variables $\mathscr{L}$ becomes

$$\frac{M}{N}\frac{\alpha^2}{\beta^2}\sum_a \left\{\frac{1}{2}\dot{\mathbf{X}}_a^2 - \frac{n^2}{n+1} D_a^{1/n} + \frac{nQ}{8\pi N}\sum_j \frac{\operatorname{erf}(u_{ai}/h)}{u_{aj}}\right\}$$

where

$$D_a = \frac{Q}{N}\left(\frac{1}{\pi h^2}\right)^{3/2} \sum_{j=1}^{N} \exp\left(-\frac{u_{aj}^2}{h^2}\right).$$

kernel

Why use a Lagrangian ?

Software improvements

1. Improved kernels.
2. Improved rule for h.
3. Faster accessing of particles.
4. Finer control of dissipation e.g viscosity when simulating shocks.
5. Extended use of Lagrangians.
6. Methods to make Div(B) very small in MHD.

Different people will have different favourites

Kernels

Gaussian kernel

spherical polytrope. Using the new variables $\mathscr{L}$ becomes

$$\frac{M}{N}\frac{\alpha^2}{\beta^2}\sum_a \left\{ \frac{1}{2}\dot{\mathbf{X}}_a^2 - \frac{n^2}{n+1} D_a^{1/n} + \frac{nQ}{8\pi N}\sum_j \frac{\mathrm{erf}(u_{ai}/h)}{u_{aj}} \right\}$$

where

$$D_a = \frac{Q}{N}\left(\frac{1}{\pi h^2}\right)^{3/2} \sum_{j=1}^{N} \exp\left(-\frac{u_{aj}^2}{h^2}\right).$$

kernel

1. The gaussian kernel is continuous.
2. When <500 particles used
   the large range was not a problem.
3. The gravitational forces could be smoothed
   easily and consistently.

Spline kernels 1985

Finite domain

$$W_3(\mathbf{r},h)=\frac{1}{\pi h^3}\begin{cases}2\left(\frac{3}{4}-v^2\right); & 0\leqq v\leqq\frac{1}{2}\\ \left(\frac{3}{2}-v\right)^2; & \frac{1}{2}\leqq v\leqq\frac{3}{2}\\ 0; & v>\frac{3}{2},\end{cases} \tag{20}$$

$$W_4(\mathbf{r},h)=\frac{1}{\pi h^3}\begin{cases}\frac{3}{2}\left(\frac{2}{3}-v+\frac{1}{2}v^3\right): & 0\leqq v\leqq 1\\ \frac{1}{4}(2-v)^3; & 1\leqq v\leqq 2\\ 0; & v>2,\end{cases} \tag{21}$$

where $v=r/h$. The kernels are normalized so that

$$4\pi\int_0^\infty W_n(\mathbf{r},h)r^2dr=1\,.$$

Wendland Kernels

Wendland 1995

JJM 2005   Dehnen and Aly 2012

An example in two dimensions

q = r/h

$$W(q) = \frac{7}{64\pi h^2}(2-q)^4(1+2q)$$

An example in three dimensions

$$W(q) = \frac{21}{256\pi h^3}(2-q)^4(1+2q)$$

Super kernels

Designed to remove higher order errors due to smoothing. They are corrupted by errors from approximating integrals by summation.

1979 $$W(q) = \frac{1}{\pi^{3/2}h^3} e^{-q^2} \left(\frac{5}{2} - q^2 \right)$$

can be negative

1980 $$W(q) = \frac{4}{\pi^{3/2}h^3} e^{-q^2} (1 - \frac{3}{8}\sqrt{\pi} q)$$

this repels because derivative does not vanish at q = 0

2015 Hu + Adams

Combination of super and ordinary kernels

$$W(q) = \theta W_s(q) + (1-\theta) W_o(q)$$

The super kernels can be negative

switch to ordinary near shocks
switch to super otherwise

The problem is how to make a good switch

Different kernels can be used for different processes.

1. Viscous terms.
2. Thermal terms.
3. Drag terms for dusty gas

An example.
A kernel for the viscous term 1988

$$\frac{\alpha hmc}{\rho}\sum_k (v_j - v_k)\frac{\partial W}{u \partial u}$$

One dimension

if there are sinusoidal oscillations

$$-i\omega\alpha hc \sum_j a(1 - e^{-iK(j-k)})\frac{\partial W}{u \partial u}$$

the continuum limit with a Gaussian kernel

$$\frac{i\omega ca2}{h}(1 - e^{-K^2/4}) \quad \to K^2$$

as K goes to zero

By choosing a super kernel

$$W = \frac{1}{2\sqrt{\pi}} \left(2e^{-u^2} - \sqrt{2} e^{-u^2/2} \right)$$

you get a dissipation term

$$\left(1 - e^{-K^2/4}\right)^2 \quad \rightarrow K^4$$

my notebook contains several pages of discussion of different kernels in 1D and 2D to reduce viscous effects.

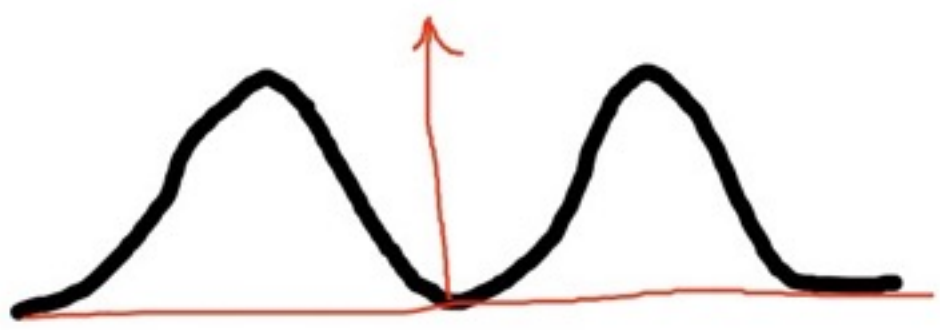
Nothing was published.

Drag terms for a dusty gas

(Laibe and Price)

Example

$$W(q) \propto q^2 e^{-q^2/h^2}$$



Also used, but without success, by Joe Morris to remove the **tensile instability**

Flexibility

The use of different kernels for
different processes is
equivalent
to a finite difference scheme
with
different grids for different processes.
This is very difficult to do
so no one does it!

Calculating the gravitational field

Finding the gravitational field in the early days.

$$\nabla^2 \Phi = 4\pi G \rho$$

substitute

$$\rho(\mathbf{r}) = \left(\frac{1}{\pi h^2}\right)^{3/2} \sum_b m_b e^{-(\mathbf{r}-\mathbf{r}_a)^2/h^2}$$

and integrate exactly.

These methods were superseded

by

1. Finite difference methods (4th order)
2. Tree codes

Resolution length h

Need to choose h proportional
to a natural length scale. For astrophysics
we thought the natural length scale came from the
gravitational energy

$$h \propto \frac{1}{grav\ energy}$$

Gravitational energy

where $f = 1/h^2$. In (2.7) the self-energy terms $(i = j)$ must be included because the particles represent distributions of matter not points. When using the Gaussian smoothing functions h occurs frequently in the form $1/h^2$. For this reason, in our calculations, we determine $1/h^2$ which, according to the above, is given by

$$\frac{1}{h^2} = A\left\{\frac{2}{N^2}\left(\frac{f}{\pi}\right)^{1/2}\sum_i\sum_j\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|}\int_0^{|\mathbf{r}_i-\mathbf{r}_j|}\exp\left(-fv^2\right)dv\right\}^2. \qquad (2.8)$$

The constant A, in the range $4 < A < 8$ gives satisfactory results when $N = 80$. For larger values of N, A can be made larger, the trend being approximately $A \propto N^{2/7}$ (GM).

h was later related to the average density

$$h \propto \frac{1}{< \rho >^{1/d}}$$

where

$$\sum_i \sum_j e^{-r_{ij}^2/h^2} = constant$$

Therefore h was constant in space but varied with time.

The reason for this is that we didn't know how to take account a spatial variation in h.

This has now been resolved for compressible flow by taking

$$\rho h^d = constant$$

then using a Lagrangian to get the equations of motion.

Daniel Price: J. Comp. Phys. vol 231,(2010)

J.J. Monaghan: Rep. Prog. Phys. vol 68, (2005)

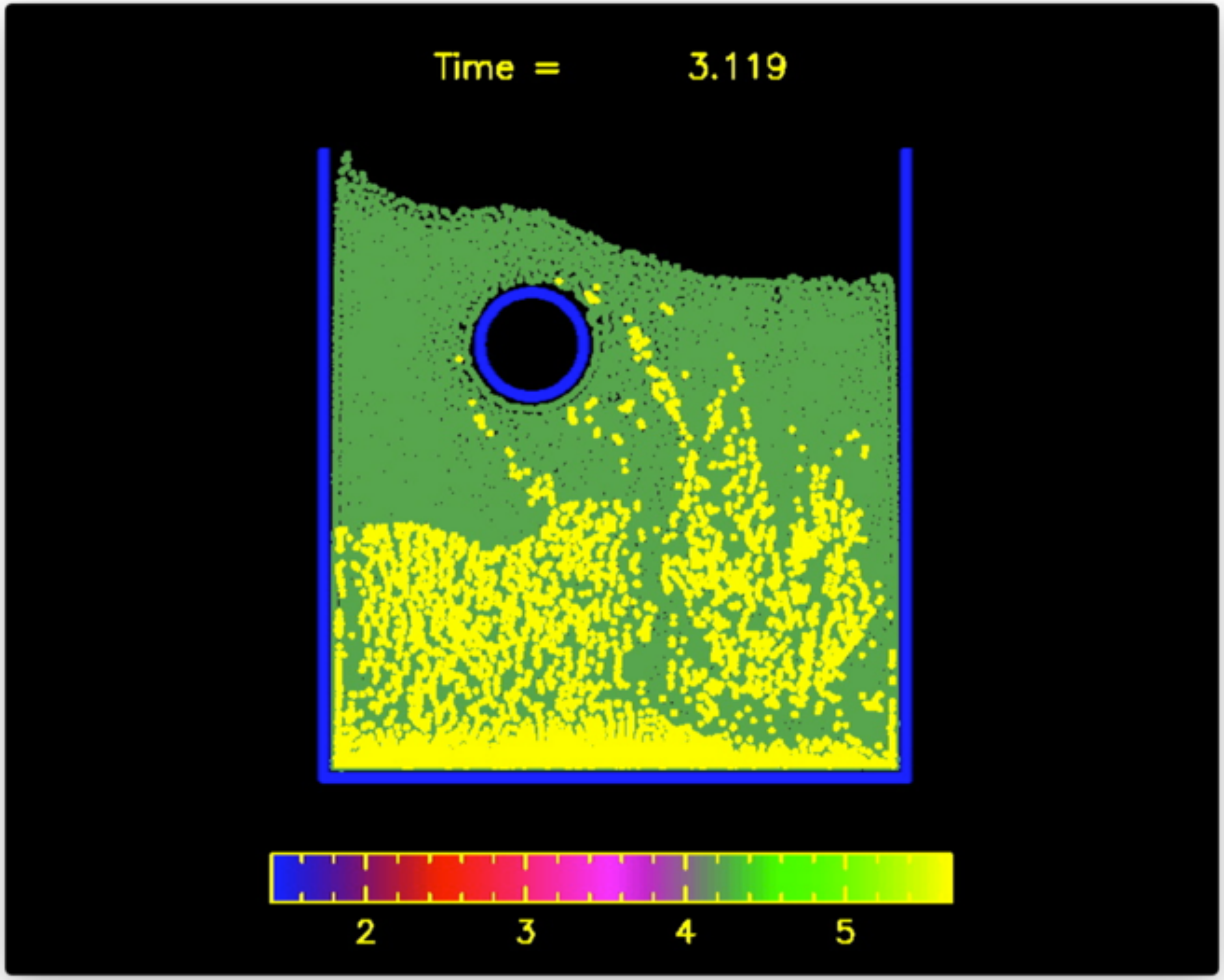
J. J. M. MNRAS 2002, Springel + Hernquist MNRAS 2002.

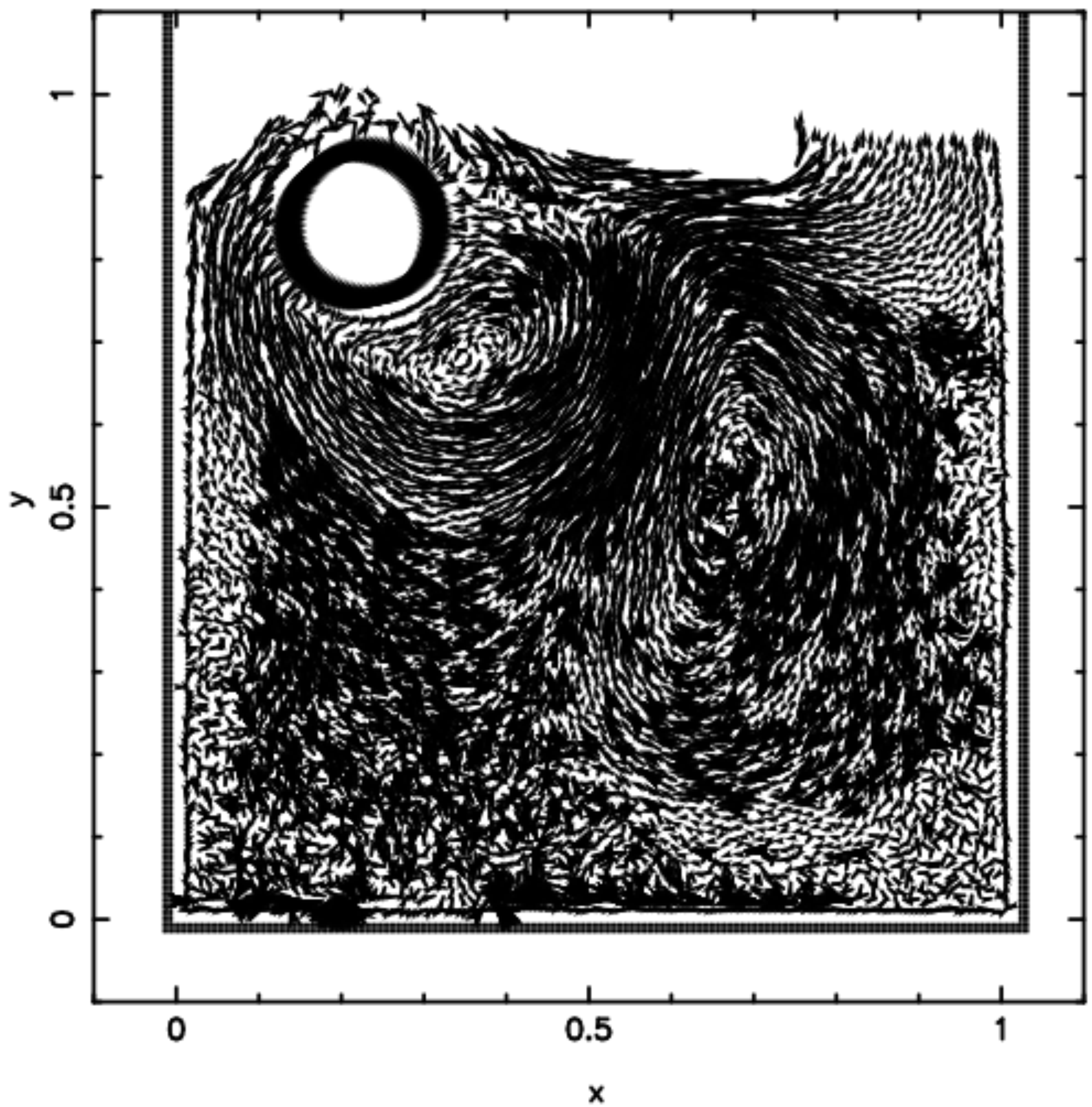
See the paper concerning
h for astrophysical problems
with magnetic fields
Ben Lewis

Variable h is even necessary with incompressible fluids if they contain dust and other particulate matter

SPH simulation of dust and water

Note the inhomogeneous SPH dust particles




0
0.5
1
x
y

Accessing particles

1. Direct summation & using symmetry
2. Link lists & Rank lists
3. Tree code

A key improvement came from Hardware which meant more particles.

Date	n
1977	**40**
1978	**80**
1983	**1000**
1986	**12,700**
1997	**93,000**
2005	$\sim 10^6$
2015	$10^6 - 10^{10}$
2025	$10^9 - 10^{12}$

CDC and Vax computers

Clusters of processors

direct summation

link list

tree code

This is approximately Moore's Law

Using these various techniques Matthew Bate simulated star formation in detail.

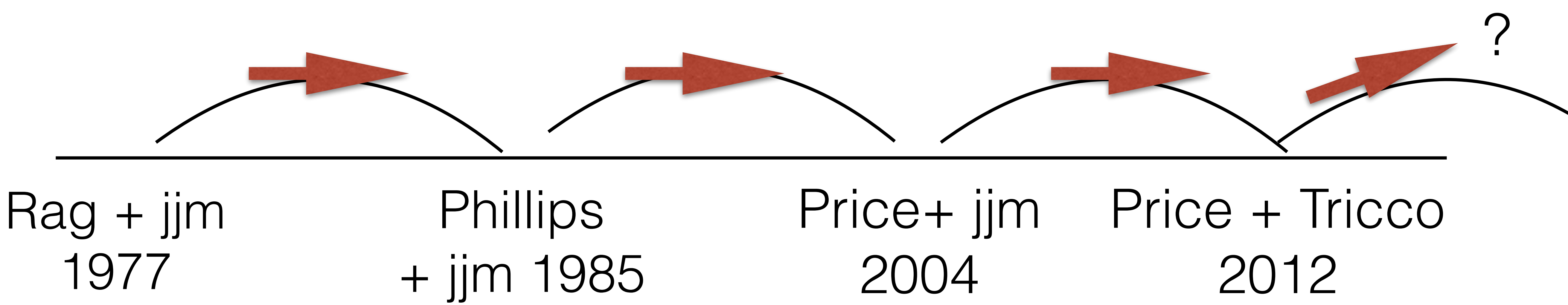
Matthew Bate - Exeter

SPH simulation of Stars forming in a large gas cloud

In astrophysics magnetic field produce important forces

SPH Magnetic field simulations

The 8 year jump

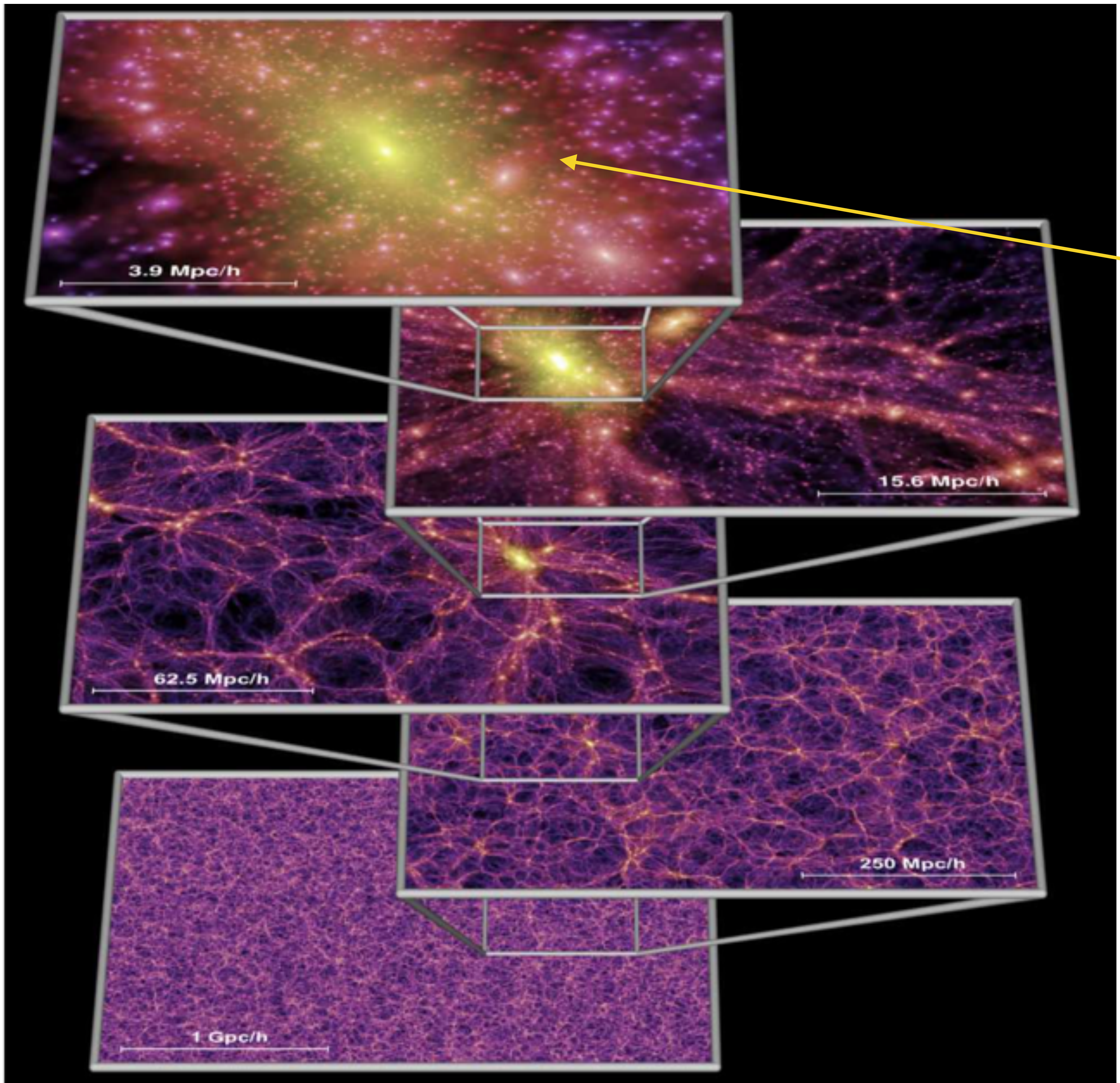


The crucial constraint

$$\nabla \cdot \mathbf{B} = 0$$

satisfied by introducing an extra equation

Springel et al. Cosmology 10^{10} dark matter SPH particles



galaxies forming

Gas and Dark matter 3×10^9 SPH particles

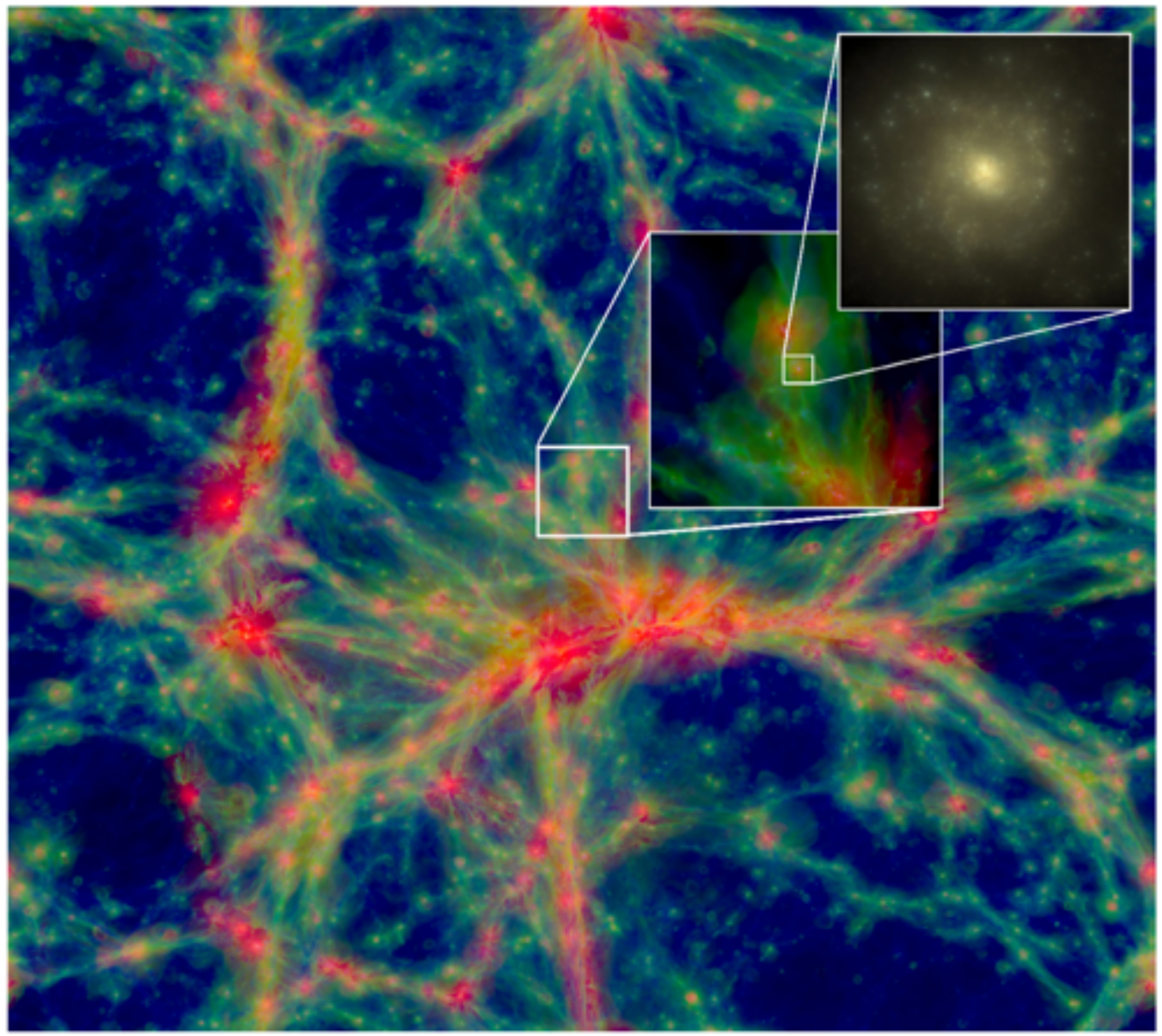


Figure 1. A 100 × 100 × 20 cMpc slice through the Ref-L100N1504 simulation at $z = 0$. The intensity shows the gas density while the colour encodes the gas temperature using different colour channels for gas with $T < 10^{4.5}$ K (blue), $10^{4.5}$ K $< T < 10^{5.5}$ K (green), and $T > 10^{5.5}$ K (red). The insets show regions

Mon. Not. Roy. Astro. Soc.
Schaye et al. 2015

SPH extended to problems involving water in 1994

The Volcanic eruption of Thera 1500 BC & its effect on the Minoans

Cliff
200m high

yachts in the Caldera

deposits from pyroclastic flows 30 m deep

Pyroclastic flow at Montserrat

T = 600 C

steam

For marine scientists and engineers
one aim was to simulate
waves on the coast or ships
in a storm.

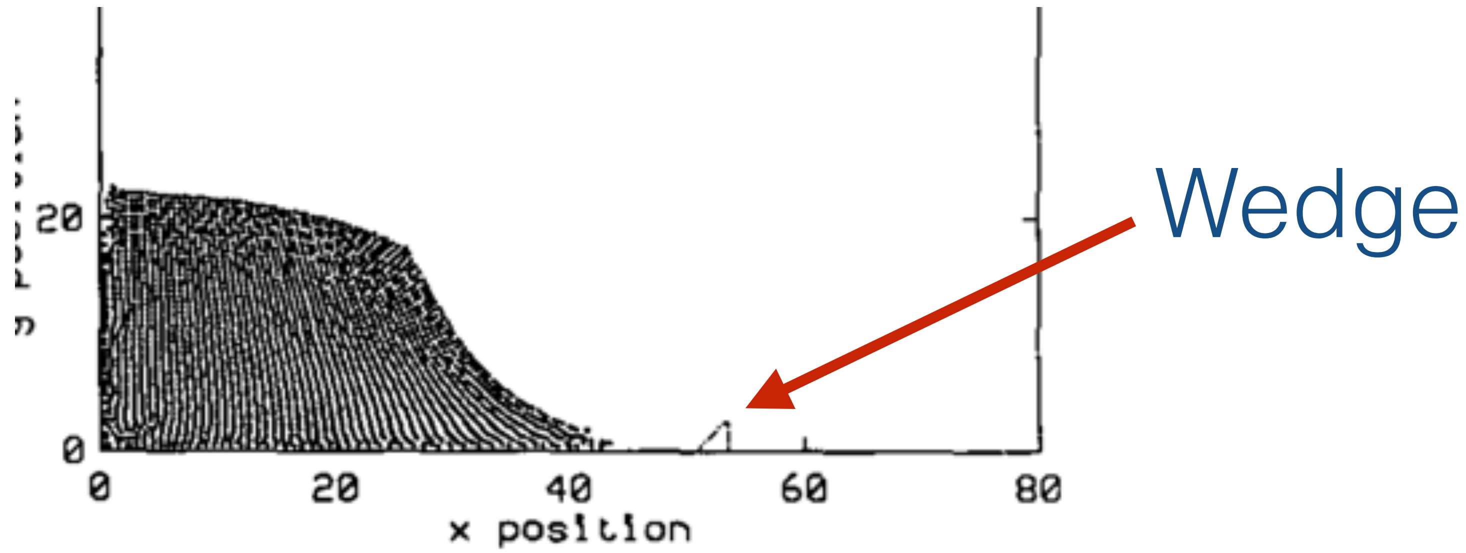
Simulating the flow of water
with and without solid bodies
began in 1994,
using the early astrophysical algorithms.

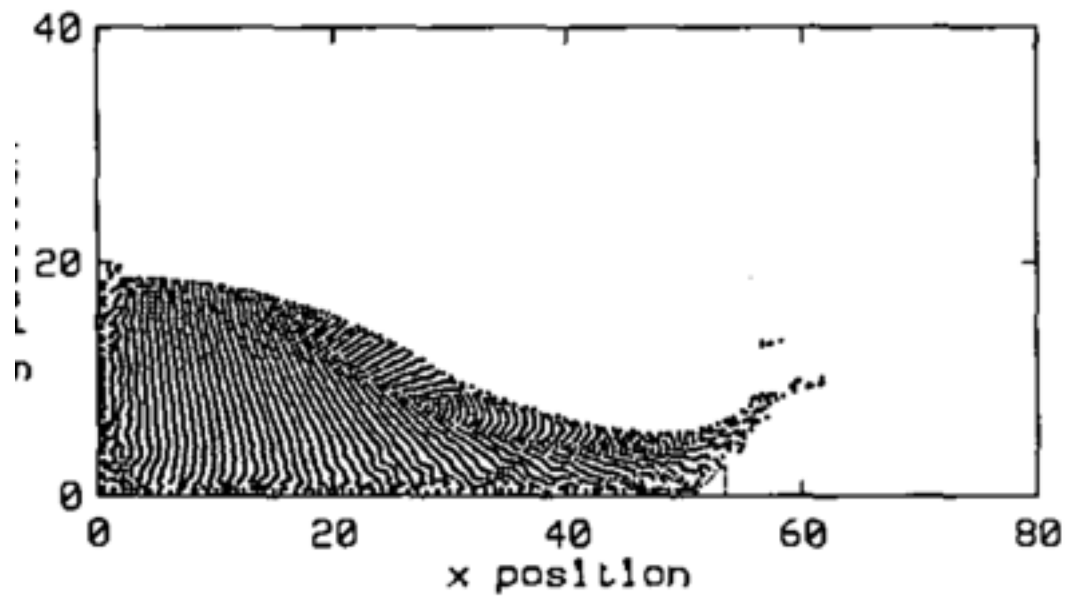
A key point was to make the fluid
weakly compressible.

$$P = B\left(\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right)$$

The first simulations were simple.

1994
2D dam break with 3000 SPH particles





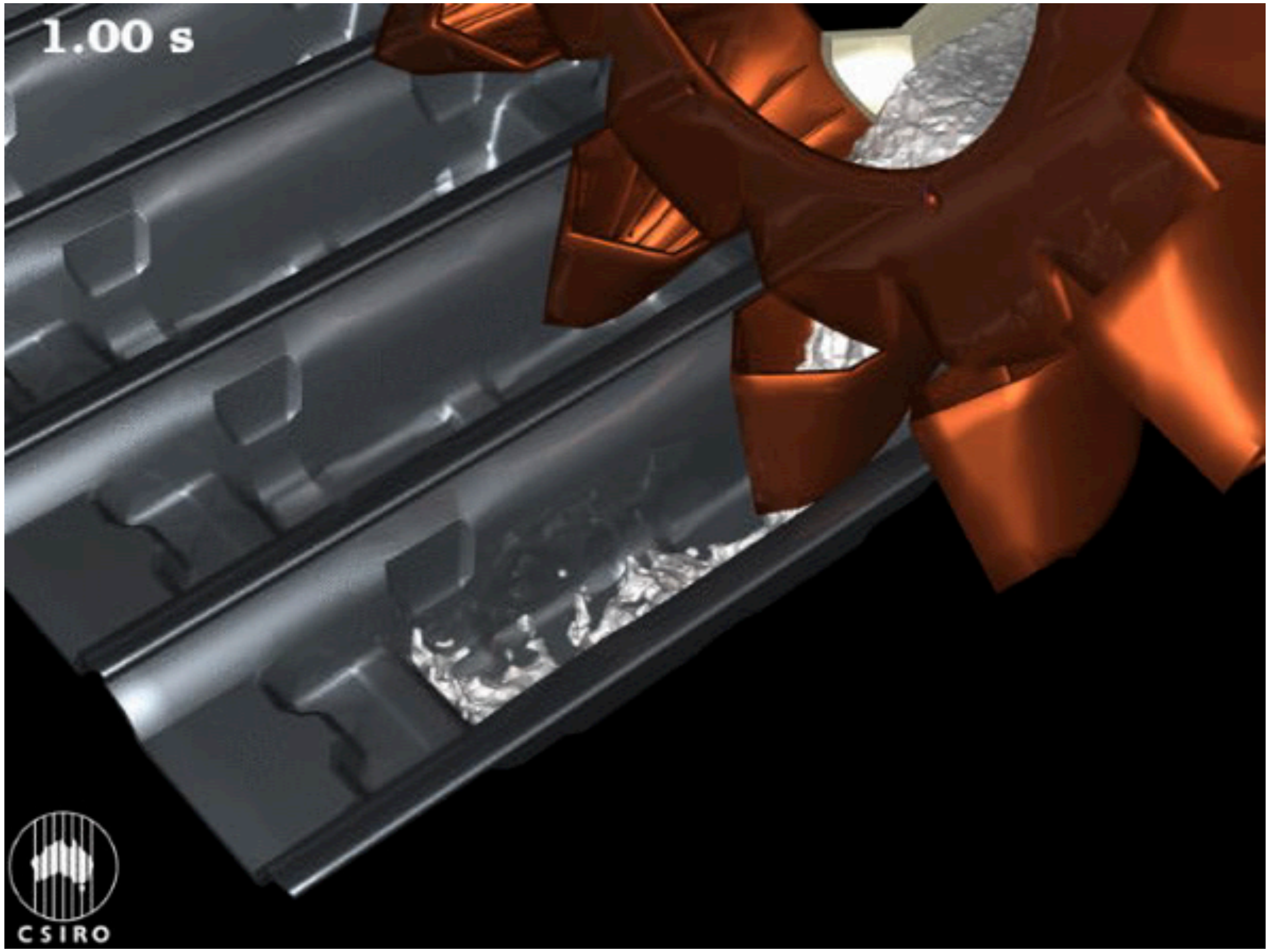
Around 2005 they became more complex.

See also the Web site for the company
Next Limit specializing in special effects.

Casting of Aluminium Ingots from a fadewheel system

CSIRO
Paul Cleary, Mahesh Prakash
Joseph Ha
based on codes from JJM

500000 liquid particles
boundaries with boundary particles
2001


1.00 s
CSIRO

Some new techniques were found such as

moving with a different v

note

XSPH 1989

$$\hat{\mathbf{v}}_a = \mathbf{v}_a + \epsilon \sum_b \frac{m_b}{\bar{\rho}_{ab}} (\mathbf{v}_b - \mathbf{v}_a) W^*_{ab}$$

note

$$\frac{d\mathbf{v}}{dt} = \mathbf{F} \qquad \frac{d\mathbf{r}}{dt} = \hat{\mathbf{v}}$$

Extended to turbulence

ϵSPH 2011

conservation

Note

$$L = \sum_b m_b \left(\frac{1}{2} \hat{\mathbf{v}}_b \cdot \mathbf{v}_b - u(\rho_b) - \Phi_a \right)$$

Conserves circulation accurately
and
truncates the spectrum

another formulation

2013
adami hu adams

correcting the acceleration using a constant background pressure

$$\hat{\mathbf{v}}_a(t+dt) = \mathbf{v}_a(t) + dt\left(\frac{d\hat{\mathbf{v}}_a}{dt} - \frac{1}{\rho_a}\nabla P_{back}\right)$$

The Future

Speeding up the code

1. Hardware improvements:

memory and cycle time and communication.

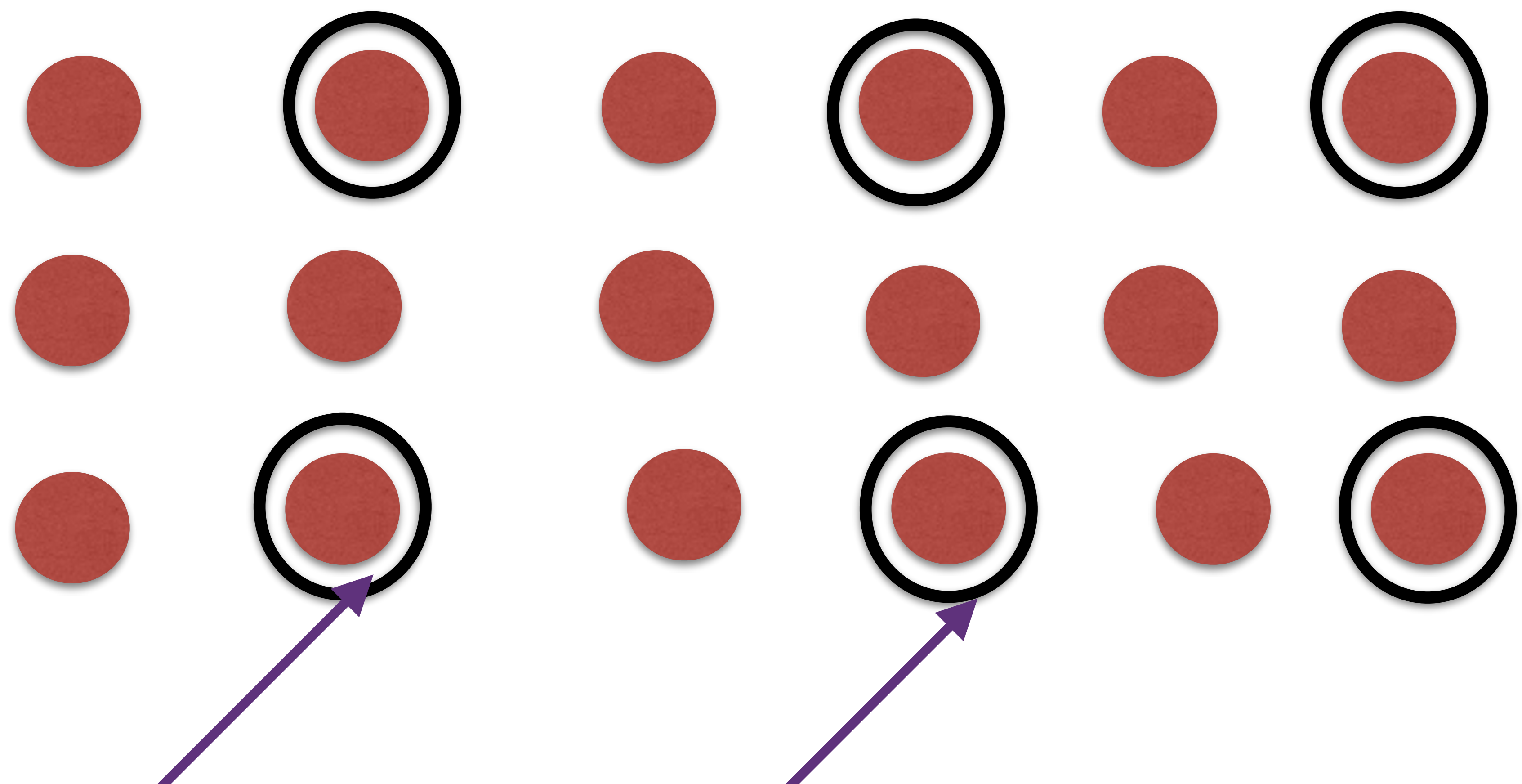
2. Possible Software improvements

1. Something like PIC with 2 sets of particles

2. Multi-set iteration Similar to Multi-grid for Heat conduction.

3. In link-list cells average properties.

PIC like scheme

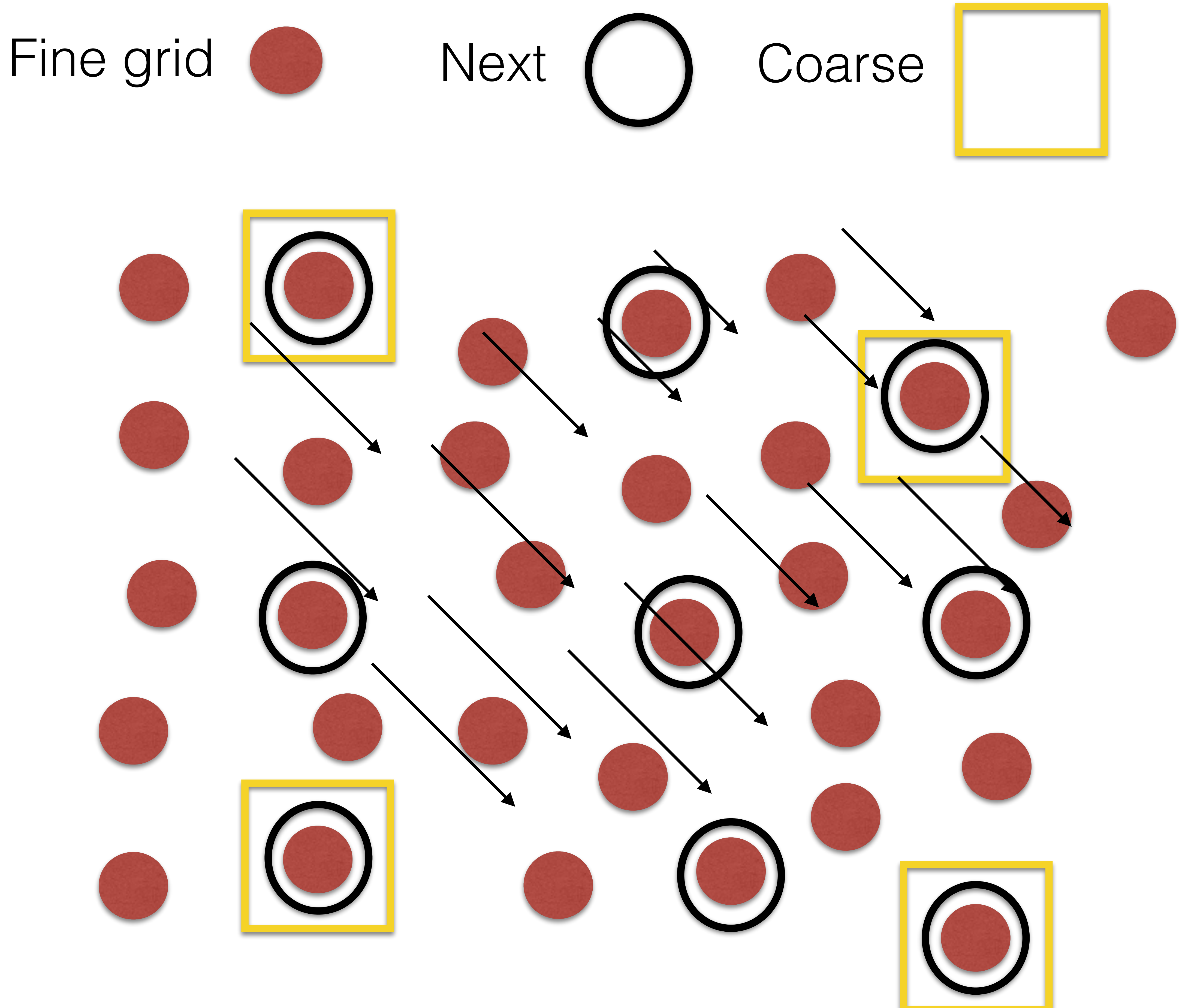


Calculate forces on the black ringed particles

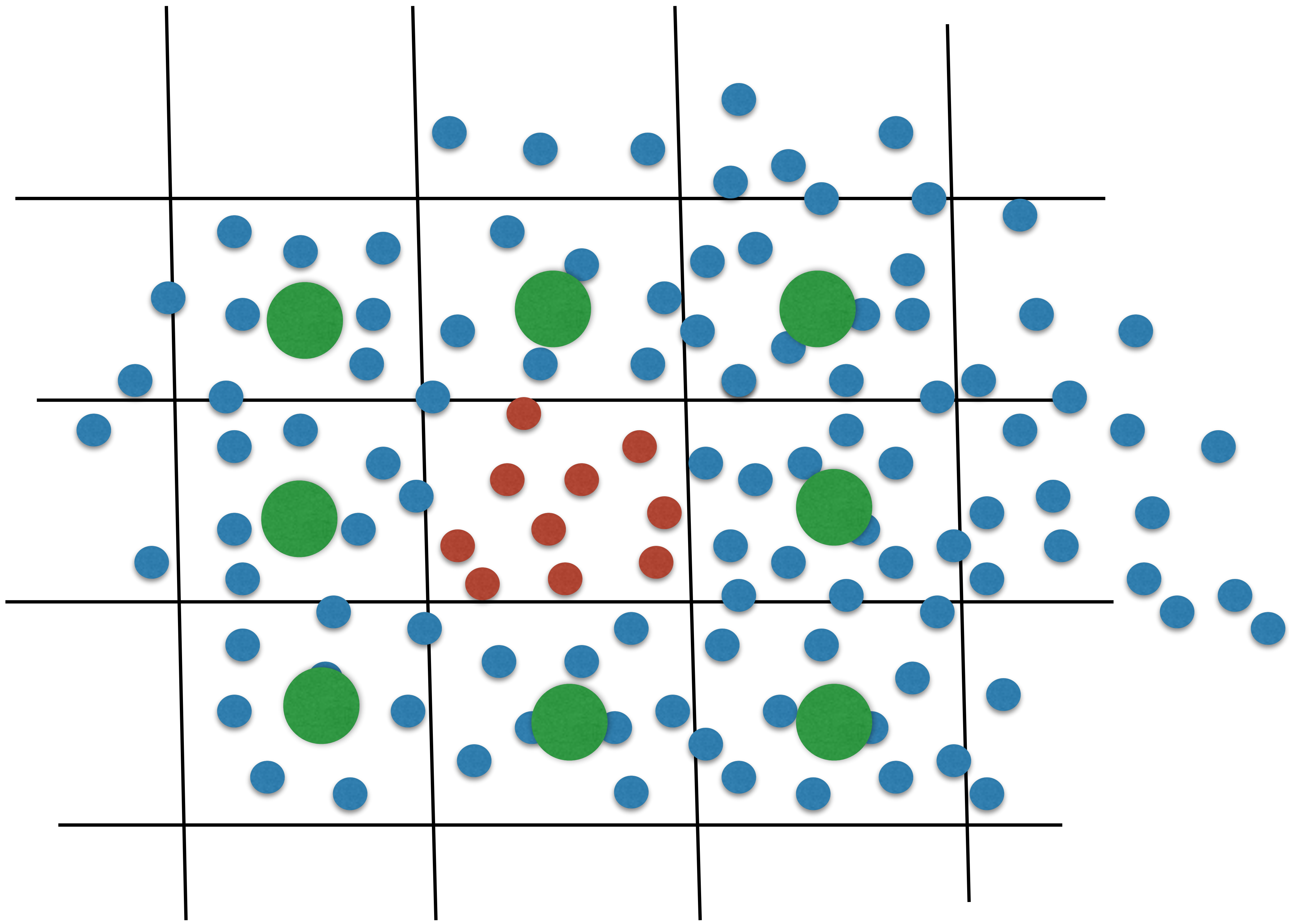
Map these forces to the other particles

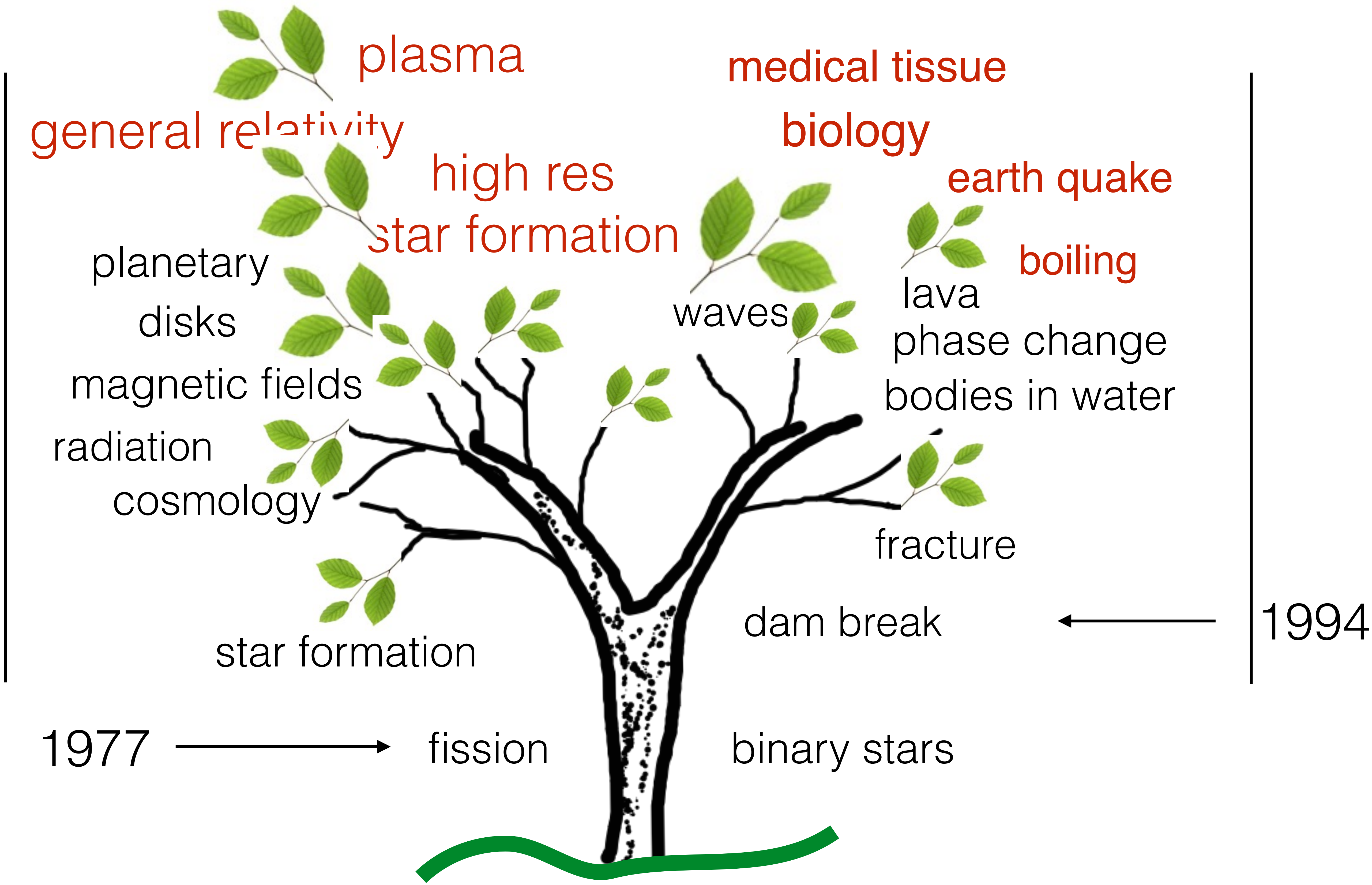
Multi-set iteration Similar to Multi-grid for Heat conduction.

Instead of grids use subsets of particles which have greater spacing and faster convergence.


Fine grid
Next
Coarse

The green denotes virtual particles





SPH tree of life grows and grows

There is much more that can be considered.

I look forward
to hearing about unexpected
and useful ideas during
the next few days.

Thank you